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Conservation laws of *high-order* nonlinear PDEs and the variational conservation laws in the class with mixed derivatives

R Narain and A H Kara

School of Mathematics, University of the Witwatersrand, Wits 2050, Johannesburg, South Africa

E-mail: Abdul.Kara@wits.ac.za

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Abstract

The construction of conserved vectors using Noether's theorem via a knowledge of a Lagrangian (or via the recently developed concept of partial Lagrangians) is well known. The formulas to determine these for higher order flows are somewhat cumbersome but peculiar and become more so as the order increases. We carry out these for a class of high-order partial differential equations from mathematical physics and then consider some specific ones with mixed derivatives. In the latter set of examples, our main focus is that the resultant conserved flows display some previously unknown interesting 'divergence properties' owing to the presence of the mixed derivatives. Overall, we consider a large class of equations of interest and construct some new conservation laws.

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1. Introduction

When considering the construction of conservation laws via Noether's theorem using a Lagrangian or a 'partial Lagrangian', an interesting situation arises when the equations under investigation are such that the highest derivative term is mixed; the mixed derivative term is the one that involves differentiation by more than one of the independent variables. When substituting the conserved flow back into the divergence relationship, a number of 'extra' terms (on which the Euler operator vanishes) arise. Thus, we have essentially 'trivial' conserved quantities that need to be fed back into the conserved vectors that are computed initially via Noether's theorem—these are necessary terms that may guarantee the notion of 'association' between conserved flows and symmetries (see [2, 13, 14])—otherwise, the total divergence of the computed conserved flows is the equations modulo the trivial part. Sometimes, there may be no association of a conservation law with a symmetry. However, the association is established by merely including a specific total divergence (trivial part). The manner in which this is carried out is defined in [16]. Thus, the 'extra' divergence terms that appear in the particular type of examples studied here are significant in establishing or rejecting a connection between a given symmetry and a conservation law.

Firstly, a variety of high-order equations are studied. For example, the fifth-order KdV and fourth-order Boussinesq equations are well-known examples from mathematical physics purported to be of 'high' order. For these and any high-order partial differential equations (PDEs), finding conservation laws by first principles can be extremely tedious. Thus, one needs to resort to alternate methods appealing to the underlying symmetry generators of the equations. If this means the variational route, then there may be problems such as the existence and determination of a Lagrangian. For the two cases cited here, we construct a 'weak' or 'partial' Lagrangian and successfully construct conservation laws. The point of emphasis is the cumbersome formulas that are required in the determination of the conserved flows due to the order of the Lagrangians and related functions.

Other examples we consider are the fourth-order shallow-water wave and regularized long-wave equations. These equations have their importance in many areas of physics, and real-world applications, e.g. tsunamis which are characterized with long periods and wavelengths; as a result they behave as shallow-water waves. Also, we study the well-known Camassa–Holms, Hunter–Saxton, inviscid Burgers and KdV family of equations—identifying the consequences of mixed highest derivative terms in the PDE.

The practical and mathematical role of conservation laws is now well established (see [4] and references therein). Firstly, the conserved vectors provide a mechanism for reducing a PDE via *potential variables* to *potential systems* and one can analyze the PDE by studying the reduced potential form [12]. In contrast, some of the other conservation laws have a physical meaning, such as conservation of linear/angular momentum and energy, and *Lorentz rotation*. Furthermore, the quantitative and qualitative properties of solutions of PDEs are established through conservation laws. For example, a numerical solution of PDEs can be checked via a knowledge of the underlying conservation laws, i.e. one could check that conserved quantities indeed remain constant. See also [5, 6, 19, 22, 24].

We present the notation and preliminaries that will be used.

Consider an rth-order system of partial differential equations of n independent variables $x = (x^1, x^2, \ldots, x^n)$ and m dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = O, \qquad \mu = 1, \ldots, \tilde{m}, \tag{1.1}$$

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first-, second-, …, rth-order partial derivatives, that is $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \qquad i = 1, \ldots, n, \tag{1.2}$$

where the summation convention is used whenever appropriate.

A current $T = (T^1, \ldots, T^n)$ is conserved if it satisfies

$$D_i T^i = 0 \tag{1.3}$$

along the solutions of (1.1).

Suppose $\mathcal{A}$ is the universal space of differential functions. A Lie–Bäcklund operator is given by

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \zeta_{i_1 i_2}^\alpha \frac{\partial}{\partial u_{i_1 i_2}^\alpha} + \cdots, \tag{1.4}$$

where $\xi^i,\ \eta^\alpha \in \mathcal{A}$ and the additional coefficients are

$$\begin{aligned}\zeta_i^\alpha &= D_i(W^\alpha) + \xi^j u_{ij}^\alpha,\\ \zeta_{i_1i_2}^\alpha &= D_{i_1}D_{i_2}(W^\alpha) + \xi^j u_{ji_1i_2}^\alpha,\\ &\ \vdots\end{aligned} \tag{1.5}$$

and W^α is the Lie characteristic function defined by

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \tag{1.6}$$

Here, we will assume that X is a Lie point operator, i.e. ξ and η are functions of x and u and are independent of derivatives of u, respectively.

The Euler–Lagrange equations, if they exist, associated with (1.1) are the system $\delta L/\delta u^\alpha = 0, \alpha = 1,\dots,m$, where $\delta/\delta u^\alpha$ is the Euler–Lagrange operator given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geqslant 1}(-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_s}}, \qquad \alpha = 1,\dots,m. \tag{1.7}$$

L is referred to as a Lagrangian and a Noether symmetry operator X of L arises from a study of the invariance properties of the associated functional

$$\mathcal{L} = \int_\Omega L(x,u,u_{(1)},\dots,u_{(r)})\,\mathrm{d}x \tag{1.8}$$

defined over Ω. If we include point-dependent gauge terms $f_1,\dots,f_n$, the Noether symmetries X are given by

$$XL + LD_i\xi^i = D_i f_i. \tag{1.9}$$

Corresponding to each X, a conserved flow is obtained via Noether's theorem.

For partial Lagrangians (see [15]), L, the Noether-type generators, X, are determined by

$$XL + LD_i\xi^i = W^\alpha\frac{\delta L}{\delta u^\alpha} + D_i f_i \tag{1.10}$$

and the conserved vector from the expression as in Noether's theorem (see [21]).

A further detailed analysis of the operators is completely given below for the scalar case in two dimensions, namely $(x^1,x^2) = (t,x)$. This discussion is peculiar to our work in the following as the Lagrangians and conserved flows are of high order (third order). The proofs and finer details of the results are obtainable in [9]. Suppose $X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \phi(t,x,u)\partial_u$ is a Noether point symmetry generator with gauge (f,g). Then the conserved flow (T^t,T^x) (or (T^1,T^2)) is given by

$$\begin{aligned}T^t &= L\tau + W\frac{\delta L}{\delta u_t} + D_t(W)\frac{\delta L}{\delta u_{tt}} + D_x(W)\frac{\delta L}{\delta u_{tx}}\\ &\quad + D_tD_t(W)\frac{\delta L}{\delta u_{ttt}} + D_tD_x(W)\frac{\delta L}{\delta u_{tx}} + D_xD_x(W)\frac{\delta L}{\delta u_{txx}}\\ &= L\tau + W\left(\frac{\partial L}{\partial u_t} - D_t\frac{\partial L}{\partial u_{tt}} - D_x\frac{\partial L}{\partial u_{tx}} + D_t^2\frac{\partial L}{\partial u_{ttt}} + D_x^2\frac{\partial L}{\partial u_{txx}} + D_tD_x\frac{\partial L}{\partial u_{ttx}}\right)\\ &\quad + D_t(W)\frac{\delta L}{\delta u_{tt}} + D_x(W)\frac{\delta L}{\delta u_{tx}} + D_tD_t(W)\frac{\delta L}{\delta u_{ttt}} + D_tD_x(W)\frac{\delta L}{\delta u_{tx}} + D_xD_x(W)\frac{\delta L}{\delta u_{txx}},\end{aligned}$$

$$T^x = L\xi + W\frac{\delta L}{\delta u_x} + D_t(W)\frac{\delta L}{\delta u_{xt}} + D_x(W)\frac{\delta L}{\delta u_{xx}}$$
$$+ D_tD_t(W)\frac{\delta L}{\delta u_{xtt}} + D_tD_x(W)\frac{\delta L}{\delta u_{xxt}} + D_xD_x(W)\frac{\delta L}{\delta u_{xxx}}$$
$$= L\xi + W\left(\frac{\partial L}{\partial u_x} - D_t\frac{\partial L}{\partial u_{xt}} - D_x\frac{\partial L}{\partial u_{xx}} + D_t^2\frac{\partial L}{\partial u_{xtt}} + D_x^2\frac{\partial L}{\partial u_{xxx}} + D_tD_x\frac{\partial L}{\partial u_{txx}}\right)$$
$$+ D_t(W)\frac{\delta L}{\delta u_{xt}} + D_x(W)\frac{\delta L}{\delta u_{xx}} + D_tD_t(W)\frac{\delta L}{\delta u_{xtt}} + D_tD_x(W)\frac{\delta L}{\delta u_{xxt}}, \tag{1.11}$$

where

$$\frac{\delta}{\delta v} = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_t^2\frac{\partial}{\partial v_{tt}} + D_x^2\frac{\partial}{\partial v_{xx}} + D_tD_x\frac{\partial}{\partial v_{tx}} - \cdots. \tag{1.12}$$

A range of literature pertaining to conservation laws is now available mainly presenting the various methods involved, see [16, 21].

2. High-order equations—illustrative

2.1. The fifth-order KdV equation

The propagation of surface waves in a shallow channel of constant depth is described by the well-known KdV equation. It is derived from the equations of hydrodynamics for an inviscid, irrotational, incompressible fluid. If one carries to the next order, one obtains an evolution equation with a fifth-order derivative, called the fifth-order KdV equation. The particular case that we investigate is the well-known generalized fifth-order KdV equation:

$$v_t + \beta/2vv_{xxx} + \alpha v_xv_{xx} + \gamma v^2v_x + v_{xxxxx} = 0, \tag{2.1}$$

which, for a variety of combinations of the parameters, has been studied using a number of methods, analytical and numerical. Inc [10] and Abbasandy and Zakaria [1] made a detailed numerical study using the Adomian decomposition and homotopy analysis methods, respectively. Several works on the soliton solutions and various analytical methods have been done. For example, Lax [19] ($\beta/2 = 10, \alpha = 20, \gamma = 30$), Sawada-Kotera [23] ($\beta/2 = 5, \alpha = 5, \gamma = 5$), Ito [11] ($\beta/2 = 3, \alpha = 6, \gamma = 2$). The well-known Kaup–Kuperschmidt equation is based on the case $\beta/2 = -15, \alpha = -15, \gamma = 45$. It can be shown that the equation is Hamiltonian for $\beta = 2\alpha$ on the principle $v_t = D_x(\delta\mathcal{H})$, where $\mathcal{H} = -\int(\alpha uu_{xx} + \alpha/2u_x^2 + \gamma/(12)u^4 + 1/2u_{xx}^2)\mathrm{d}x$.

The standard third-order KdV equation is an evolution equation, but its differential consequence admits a Lagrangian [8] and, thus, the KdV equation itself is construed as a variational equation. We show that one can do this for (2.1) by which some interesting results regarding conservation laws via Noether's theorem are obtained. This analogous study of the fifth-order KdV has not, to the knowledge of the authors, been carried out before. This may be due to the cumbersome forms of the extended Euler–Lagrange operators that need to be used. If $v = u_x$ in (2.1), we obtain the sixth-order equation

$$u_{xt} + \beta/2u_xu_{xxxx} + \alpha u_{xx}u_{xxx} + \gamma u_x^2u_{xx} + u_{xxxxxx} = 0, \tag{2.2}$$

which has a partial Lagrangian

$$L = -\left[\tfrac{1}{2}u_{xxx}^2 + \tfrac{1}{2}u_xu_t + \gamma/2u_x^4 + \beta/8u_x^2u_{xxx}\right] \tag{2.3}$$

so that

$$\frac{\delta L}{\delta u} = u_{xt} + \beta/2u_x u_{xxxx} + \gamma u_x^2 u_{xx} + u_{xxxxxx} = (\beta - \alpha)u_{xx}u_{xxx} \tag{2.4}$$

in (1.10). The separation of monomials is

$$\begin{aligned}
u_t u_x^2 u_{xxx} &: \tau_u, \\
u_t u_x^2 u_{xx} u_{xxx} &: \tau_{uu}, \\
u_x^2 u_{xx} u_{xxx} &: \xi_{uu}, \\
u_x u_{xxx}^2 &: \xi_u, \\
u_x^3 u_{xxx} &: \phi_{uuu}, \\
u_{xxx} u_{xxt} &: \tau_x, \\
u_{xxx}^2 &: \frac{5}{2}\xi_x - \frac{1}{2}\tau_t - \phi_u, \\
u_x^2 u_{xxx} &: \frac{1}{2}\left[\frac{5\beta}{4}\xi_x - \frac{1}{4}\beta(\xi_x + \tau_t) - \frac{3\beta}{4}\phi_u - 6\phi_{xuu}\right], \\
u_x u_{xx} u_{xxx} &: (-\alpha + \beta)\xi - 3\phi_{uu}, \\
u_t u_{xx} u_{xxx} &: (-\alpha + \beta)\tau, \\
u_{xx} u_{xxx} &: -(-\alpha + \beta)\phi + 3\xi_{xx} - 3\phi_{xu}, \\
u_x u_{xxx} &: -\frac{3}{4}\beta\phi_x\phi + \xi_{xxx} - 3\phi_{xxu}, \\
u_{xxx} &: -\phi_{xxx}, \\
u_x^3 u_{xx} &: -\frac{3}{8}\beta\phi_{uu}, \\
u_x^2 u_{xx} &: \frac{1}{2}\left(\frac{3}{4}\beta\xi_{xx} - \frac{3}{4}b\phi_{xu}\right), \\
u_x^4 &: \frac{9}{4}\xi_x - \frac{9}{12}\tau_t - \frac{9}{3}\phi_u, \\
u_x^3 &: -\frac{9}{3}\phi_u + \frac{1}{8}\beta\xi_{xxx}, \\
u_x^2 &: \xi_t, \\
u_t u_x &: \frac{1}{2}\xi_x + \frac{1}{2}(-\xi_x - \tau_t) + \frac{1}{2}\tau_t - \phi_u, \\
u_t &: -f_u - \frac{1}{2}\phi_x, \\
u_x &: -g_u - \frac{1}{2}\phi_t, \\
1 &: -f_t - g_x.
\end{aligned} \tag{2.5}$$

This leads to a nontrivial solution only if $\alpha = \beta$. That is, the partial Lagrangian is, in fact, a Lagrangian of (2.2) and the generators are the corresponding Noether symmetries, namely

$$\partial_t \quad (W = -u_t), \qquad \xi(t)\partial_x \quad (W = -\xi u_x). \tag{2.6}$$

We now list the corresponding conserved vectors.

(i) $\partial_t \quad (W = -u_t)$

$$T^t = -\left[\frac{1}{2}u_{xxx}^2 + \frac{1}{2}u_x u_t + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx}\right](1) + [-u_t]\left[-\frac{1}{2}u_x\right],$$

$$= -\frac{1}{2}u_{xxx}^2 - \frac{1}{2}u_x u_t - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2 u_{xxx} + \frac{1}{2}u_x u_t,$$

$$= -\frac{1}{2}u_{xxx}^2 - \frac{\gamma}{12}u_x^4 - \frac{\beta}{8}u_x^2 u_{xxx},$$

$$T^x = [-u_t]\left[-\frac{1}{2}u_t - \frac{\gamma}{3}u_x^3 - \frac{\beta}{4}u_x u_{xxx} + \mathrm{D}_x^2\left(-u_{xxx} - \frac{\beta}{8}u_x^2\right)\right]$$

$$+ \mathrm{D}_x(-u_t)\left[-\mathrm{D}_x\left(-u_{xxx} - \frac{\beta}{8}u_x^2\right)\right] + \mathrm{D}_x^2(-u_t)\left[-u_{xxx} - \frac{\beta}{6}u_x^2\right],$$

$$= (u_t)\left[\frac{1}{2}u_t + u_{xxxxx} + \mathrm{D}_x^2\left(\frac{\beta}{8}u_x^2\right)\right] + (-u_{tx})\left[u_{xxxx} + \frac{\beta}{4}u_x u_{xx}\right]$$

$$+ (u_{txx})\left[u_{xxx} + \frac{\beta}{8}u_x^2\right]$$

$$= (u_t)\left[\frac{1}{2}u_t + u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_x u_{xxx}\right] + (-u_{tx})\left[u_{xxxx} + \frac{\beta}{4}u_x u_{xx}\right]$$

$$+ (u_{txx})\left[u_{xxx} + \frac{\beta}{8}u_x^2\right].$$

Thus, $D_t T^t + D_x T^x = u_t\left[u_{xt} + \beta/2 u_x u_{xxxx} + \beta u_{xx} u_{xxx} + \gamma u_x^2 u_{xx} + u_{xxxxxx}\right] = 0.$

(ii) $\xi(t)\partial_x \quad (W = -\xi u_x)$

$$T^t = -\xi u_x\left[-\frac{1}{2}u_x\right],$$

$$= \frac{1}{2}\xi u_x^2,$$

$$T^x = -\xi\left[\frac{1}{2}u_{xxx}^2 + \frac{\gamma}{12}u_x^4 + \frac{\beta}{8}u_x^2 u_{xxx}\right] + \xi u_x\left[u_{xxxxx} + \frac{\beta}{4}u_{xx}^2 + \frac{\beta}{4}u_x u_{xxx}\right]$$

$$- \xi u_{xx}\left[u_{xxxx} + \frac{\beta}{4}u_x u_{xx}\right] + \xi u_{xxx}\left[u_{xxx} + \frac{\beta}{8}u_x^2\right] + \xi u_x\left[\frac{\gamma}{3}u_{xxx}^3 + \frac{\gamma}{4}u_x u_{xxx}\right].$$

Thus, $D_t T^t + D_x T^x = \xi(t)u_x\left[u_{xt} + \beta/2 u_x u_{xxxx} + \beta u_{xx} u_{xxx} + \gamma u_x^2 u_{xx} + u_{xxxxxx}\right] = 0.$

Remark. The conserved vector in (i) is of 'nonlocal' type for the fifth-order KdV equation (2.1) when we substitute back to v since, if $v = u_x$, $u_t = \int v_t \mathrm{d}x$.

2.2. *The fourth-order Boussinesq equation*

The family of Boussinesq equations describing the bidirectional propagation of waves in shallow water (or the behavior of long waves) is sometimes written as the fourth-order equation:

$$u_{xxxx} + uu_{xx} + u_x^2 + u_{tt} = 0. \tag{2.7}$$

Its Noether-type symmetries, $X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \phi(t,x,u)\partial_u$, via the partial Lagrangian $L = \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_t^2 - \frac{1}{2}uu_x^2$ $\left(\frac{\delta L}{\delta u} = -\frac{1}{2}u_x^2\right)$ are determined by (1.10). In this case, XL is a second prolongation of X, namely

$$\begin{aligned} XL = {} & -\tfrac{1}{2}\phi u_x^2 + u_t u_x \xi_t + u_t^2 u_x \xi_u + uu_x^3\xi_u + uu_x^2\xi_x + u_t^2\tau_t + u_t^3\tau_u + uu_t u_x^2\tau_u + uu_t u_x \tau_x \\ & - u_t\phi_t - u_t^2\phi_u - uu_x^2\phi_u - uu_x\phi_x - 2u_x\tau_u u_{x,t}u_{x,x} - 2\tau_x u_{x,t}u_{x,x} - 3u_x\xi_u u_{x,x}^2 \\ & - 2\xi_x u_{x,x}^2 - u_t\tau_u u_{x,x}^2 + \phi_u u_{x,x}^2 - u_x^3 u_{x,x}\xi_{u,u} - 2u_x^2 u_{x,x}\xi_{x,u} - u_x u_{x,x}\xi_{x,x} \\ & - u_t u_x^2 u_{x,x}\tau_{u,u} - 2u_t u_x u_{x,x}\tau_{x,u} - u_t u_{x,x}\tau_{x,x} + u_x^2 u_{x,x}\phi_{u,u} + 2u_x u_{x,x}\phi_{x,u} \\ & + u_{x,x}\phi_{x,x}, \end{aligned} \qquad (2.8)$$

which is substituted into (1.10). Separation by monomials then lead to

$$\begin{aligned} u_t u_{xx} u_x^2 &: -\tau_{uu}, \\ u_t u_{xx} u_x &: -2\tau_{xu}, \\ u_{xx}u_x^3 &: -\xi_{uu}, \\ u_{xx}u_x^2 &: -2\xi_{xu} + \phi_{uu}, \\ u_{xx}u_x &: -\xi_{xx} + 2\phi_{xu}, \\ u_{xx}u_{xt} &: \tau_x, \\ u_{xx}^2 u_x &: \xi_u, \\ u_{xx}^2 &: -\tfrac{3}{2}\xi_x + \tfrac{1}{2}\tau_t + \phi_u, \\ u_{xx} &: \phi_{xx}, \\ u_t u_x^2 &: -\tfrac{1}{2}\tau + \tfrac{1}{2}u\tau_u, \\ u_t u_x &: \xi_t, \\ u_t^3 &: \tfrac{1}{2}\tau_u, \\ u_x^3 &: -\tfrac{1}{2}\xi + \tfrac{1}{2}u\xi_u, \\ u_x^2 &: \tfrac{1}{2}u\xi_x - \tfrac{1}{2}u\tau_t - u\phi_u, \\ u_t &: -f_u - \phi_t, \\ u_x &: -g_u - u\phi_x, \\ 1 &: -f_t - g_x. \end{aligned} \qquad (2.9)$$

The overdetermined system has solution

$$\begin{aligned} &\tau = 0, \qquad \xi = 0, \qquad \phi = A + Bt + Cx + Dxt, \\ &f = -(B + Dx)u + a(x,t), \qquad g = -\tfrac{1}{2}(C + Dt)u^2 + b(x,t) \end{aligned} \qquad (2.10)$$

where $a_t + b_x = 0$ and A, B, C and D are arbitrary constants. If we choose, for example, $A = D = 0$ (Noether-type symmetry $X = (Bt + Cx)\partial_u$, $W = (Bt + Cx)$, $f = -Bu$ and $g = -\frac{1}{2}Cu^2$), we obtain, via a truncated version of (1.11), i.e.,

$$\begin{aligned} T^t &= L\tau + W\frac{\partial L}{\partial u_t} + [D_j W - WD_j]\frac{\partial L}{\partial u_{tj}} - f, \\ T^x &= L\xi + W\frac{\partial L}{\partial u_x} + [D_j W - WD_j]\frac{\partial L}{\partial u_{xj}} - g, \end{aligned} \qquad (2.11)$$

the conserved density and flux

$$\begin{aligned} T^t &= -(Bt + Cx)u_t + Bu \\ T^x &= -(Bt + Cx)uu_x + Cu_{xx} - (Bt + Cx)u_{xxx} + \tfrac{1}{2}Cu^2. \end{aligned} \tag{2.12}$$

so that $D_t T^t + D_x T^x = -(Bt + Cx)\big(u_{xxxx} + uu_{xx} + u_x^2 + u_{tt}\big)$.

3. Applications 2—mixed derivative case

3.1. Camassa–Holms, Hunter–Saxton, inviscid Burgers and KdV family of equations

The Camassa–Holms and Hunter–Saxton equations correspond to the equations of the geodesic flow with respect to different right-invariant Riemannian metrics on this group or on an associated homogeneous space [16]. Alternatively stated, the geometric interpretation of the Camassa–Holm equation is a geodesic flow equation on the group of diffeomorphisms, preserving the H1 right-invariant metric. Also, the Hunter–Saxton equation describes the propagation of waves in a massive director field of a nematic liquid crystal.

We now consider the family of equations

$$\alpha(v_t + 3vv_x) - \beta(v_{txx} + 2v_x v_{xx} + vv_{xxx}) - \gamma v_{xxx} = 0. \tag{3.1}$$

Even though it represents a class of nonlinear evolution equations, it displays variational/Hamiltonian properties and would then be subject to, amongst other things, Noether's theorem [21]. This is well documented in the case of the KdV equation [8]. Also, it displays interesting soliton or soliton-like solutions. Equation (3.1) represents a version of the KdV equation ($\alpha = 1, \beta = 0$), the Camassa–Holm equation ($\alpha = 1, \beta = 1$), the Hunter–Saxton equation ($\alpha = 0, \beta = 1$) and the inviscid Burgers equation $u_t + 3uu_x = 0$ [3, 7, 18]. We modify this equation by letting $v = u_x$ to obtain

$$\alpha(u_{tx} + 3u_x u_{xx}) - \beta(u_{txxx} + 2u_{xx}u_{xxx} + u_x u_{xxxx}) - \gamma u_{xxxx} = 0. \tag{3.2}$$

Equation (3.2) displays variational properties with respect to the Lagrangian

$$L = -\frac{\alpha}{2}\big(u_x u_t + u_x^3\big) - \frac{\beta}{2}\big(u_x u_{xx}^2 + u_{tx}u_{xx}\big) - \frac{\gamma}{2}u_{xx}^2. \tag{3.3}$$

The symmetries and corresponding conserved vectors are as follows.

(i) $X = \partial_t, \quad W = -u_t$

The conserved quantities are $T^1 = -\frac{\alpha}{2}\big(u_t u_x + u_x^3\big) - \frac{\beta}{2}\big(u_x u_{xx}^2 + u_{tx}u_{xx}\big) - \frac{\gamma}{2}u_{xx}^2 + (-u_t)\big(-\frac{\alpha}{2}u_x + \frac{\beta}{2}u_{xxx}\big) + (-u_{tx})\big(-\frac{\beta}{2}u_{xx}\big)$ and $T^2 = (-u_t)\big(-\frac{\alpha}{2}u_t - \frac{3\alpha}{2}u_x^2 + \frac{\beta}{2}u_{xx}^2 + \beta u_{txx} + \beta u_x u_{xxx} + \gamma u_{xxx}\big) + (-u_{tt})\big(-\frac{\beta}{2}u_{xx}\big) + (-u_{xx})\big(-\beta u_x u_{xx} - \frac{\beta}{2}u_{tx} - \gamma u_{xx}\big)$.

The total divergence is

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= 2\gamma u_{xx}u_{xxx} - \gamma u_{tx}u_{xxx} - \gamma u_{xx}u_{txx} + \frac{1}{2}\beta u_{xx}u_{txx} \\ &\quad + \frac{1}{2}\beta u_{tx}u_{xxx} - \beta u_{tx}u_{txx} - \frac{\beta}{2}u_t u_{txxx} + 2\beta u_x u_{xx}u_{xxx} + \beta u_{xx}^3 \\ &\quad - \beta u_x u_{xx}u_{txx} - \beta u_x u_{tx}u_{xxx} + \frac{\beta}{2}u_{xx}u_{ttx} - \frac{\beta}{2}u_{tx}u_{txx}. \end{aligned} \tag{3.4}$$

As before, extra terms that require further analysis emerge. By making an adjustment to these terms, they can be absorbed into the conservation law if we note that

$$D_t(T^1) + D_x(T^2) = D_x\left(\gamma u_{xx}^2\right) - D_x(\gamma u_{tx}u_{xx}) + D_x\left(\frac{\beta}{2}u_{tx}u_{xx}\right)$$
$$-D_x\left(\frac{\beta}{2}u_t u_{txx}\right) + D_x\left(\beta u_x u_{xx}^2\right) - D_x(u_x u_{tx}u_{xx}) \tag{3.5}$$
$$-D_x\left(\frac{\beta}{2}u_{tx}^2\right) + D_t\left(\frac{\beta}{2}u_{tx}u_{xx}\right).$$

Then by taking these differentials across and adding them to the conserved flows, this satisfies the conservation law. The modified conserved quantities are now labeled $\tilde{T}^i$, where $D_t(\tilde{T}^1) + D_x(\tilde{T}^2) = 0$ along the equation, namely

$$\tilde{T}^1 = T^1 - \frac{\beta}{2}u_{tx}u_{xx},$$
$$\tilde{T}^2 = T^2 - \gamma u_{xx}^2 + \gamma u_{tx}u_{xx} - \frac{\beta}{2}u_{tx}u_{xx} + \frac{\beta}{2}u_t u_{txx} - \beta u_x u_{xx}^2 - u_x u_{tx}u_{xx} + \frac{\beta}{2}u_{tx}^2. \tag{3.6}$$

The same applies to the following cases.

(ii) $X = \partial_x, \quad W = -u_x$

With $T^1 = (-u_x)\left(-\frac{\alpha}{2}u_x + \frac{\beta}{2}u_{xxx}\right) + (-u_{xx})\left(-\frac{\beta}{2}u_{xx}\right)$ and $T^2 = -\frac{\alpha}{2}\left(u_x u_t + u_x^3\right) - \frac{\beta}{2}\left(u_x u_{xx}^2 + u_{tx}u_{xx}\right) - \frac{\gamma}{2}u_{xx}^2 + (-u_x)\left(-\frac{\alpha}{2}u_t - \frac{3\alpha}{2}u_x^2 + \frac{\beta}{2}u_{xx}^2 + \beta u_{txx} + \beta u_x u_{xxx} + \gamma u_{xxx}\right) + (-u_{tx})\left(-\frac{\beta}{2}u_{xx}\right) + (-u_{xx})\left(-\beta u_x u_{xx} - \frac{\beta}{2}u_{tx} - \gamma u_{xx}\right)$ we get

$$D_t(T^1) + D_x(T^2) = -\tfrac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx}), \tag{3.7}$$

so that, since $-\frac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx})$ has derivative consequences,

$$-\tfrac{1}{2}\beta(u_x u_{txxx} - u_{xx}u_{txx}) = -\tfrac{1}{2}\beta\left(D_x\left(u_x u_{xx} - D_t\left(u_{xx}^2\right)\right)\right), \tag{3.8}$$

and a redefinition leads to

$$\tilde{T}^1 = T^1 - \tfrac{1}{2}\beta u_{xx}^2,$$
$$\tilde{T}^2 = T^2 + \tfrac{1}{2}\beta u_x u_{xx}. \tag{3.9}$$

(iii) $X = n(t)\partial_u, \quad W = n(t)$

Here, we get $T^1 = (n(t))\left(-\frac{\alpha}{2}u_x + \frac{\beta}{2}u_{xxx}\right)$ and $T^2 = (n(t))\left(-\frac{\alpha}{2}\left(u_x u_t + u_x^3\right) - \frac{\beta}{2}\left(u_x u_{xx}^2 + u_{tx}u_{xx}\right) - \frac{\gamma}{2}u_{xx}^2\right) + (n_t(t))\left(-\frac{\beta}{2}u_{xx}\right) + \frac{\alpha}{2}n_t(t)u,$ so that

$$D_t(T^1) + D_x(T^2) = -\tfrac{1}{2}n(t)\beta u_{txxx}, \tag{3.10}$$

and

$$\tilde{T}_2^1 = T^1,$$
$$\tilde{T}_2^2 = T^2 + \tfrac{1}{2}n(t)\beta u_{txx}. \tag{3.11}$$

3.2. *The shallow-water wave equation*

The shallow-water wave equation (SWW) models basic water waves that reasonably approximate the behavior of real ocean waves, namely

$$u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0, \tag{3.12}$$

where α and β are arbitrary constants. From equation (3.12), we separate the cases, (1) $\alpha \neq \beta$ and (2) $\alpha = \beta$.

Case (1) $\alpha \neq \beta$ will be referred to as shallow-water wave-1 (SSW-1), and corresponding to case (2) $\alpha = \beta$, in (3.12), α is replaced by β, and referred to as the shallow-water wave-2 (SSW-2), namely

$$u_{xxxt} + \beta u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx} = 0. \tag{3.13}$$

3.2.1. Shallow-water wave-1 (SSW-1). Here, we use the partial Lagrangian

$$L = \tfrac{1}{2}u_{tx}u_{xx} + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x u_t - \tfrac{1}{2}\beta u_t u_x^2, \tag{3.14}$$

for which

$$\frac{\delta L}{\delta u} = (2\beta - \alpha)u_{tx}u_x. \tag{3.15}$$

Substituting into (1.10) and separating by monomials, we obtain the system

$$\begin{aligned}
u_x u_{tx}^2 &: \tau_u, \\
u_{tx}^2 &: \tau_x, \\
u_x u_{xx}^2 &: \xi_u, \\
u_{xx}^2 &: \xi_t, \\
u_{tx} u_{xx} &: \eta_u - \xi_x, \\
u_t u_x u_{tx} &: (2\beta - \alpha)\tau, \\
u_x^2 u_{tx} &: (2\beta - \alpha)\xi, \\
u_x u_{tx} &: (2\beta - \alpha)\eta, \\
u_t u_x^2 &: \xi_x - 3\eta_u, \\
u_t u_x &: \eta_u - \beta\eta_x, \\
u_x^2 &: \eta_u - \tfrac{1}{2}\beta\eta_t - \tfrac{1}{2}\xi_x + \tfrac{1}{2}\tau_t, \\
u_x &: -g_u + \tfrac{1}{2}\eta_t + \eta_x, \\
u_t &: -f_u + \tfrac{1}{2}\eta_x, \\
1 &: f_t + g_x.
\end{aligned} \tag{3.16}$$

From equation (3.16), we observe that there are two cases that emerge: (a) $\alpha = 2\beta$ and (b) $\alpha \neq 2\beta$.

Subcase (a): $\alpha = 2\beta$ leads to the following generators and conserved vectors.

(i) $X = \partial_t, \quad W = -u_t$

The conserved flow is given by $T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$ and $T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt}$. The divergence becomes

$$\begin{aligned}
D_t(T^1) + D_x(T^2) &= D_t\left(\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_{xxx}\right) \\
&\quad + D_x\left(-u_t u_x - \tfrac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2 - \tfrac{1}{2}u_{xx}u_{tt}\right), \\
&= u_x u_{tx} + \tfrac{1}{2}u_{xxx}u_{tt} + \tfrac{1}{2}u_t u_{xxxt} - u_x u_{tx} - u_t u_{xx} - \tfrac{1}{2}u_{xx}u_{ttx} \\
&\quad - u_t u_{tx} + 2\beta u_t u_x u_{tx} + \beta u_t^2 u_{xx} + u_t u_{xxxt} - \tfrac{1}{2}u_{xxx}u_{tt}, \\
&= u_t(u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx}) + \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}, \\
&= \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}.
\end{aligned} \tag{3.17}$$

We observe that extra terms emerge. By some adjustments, these terms can be absorbed into the conservation law. That is,

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= \tfrac{1}{2}u_t u_{xxxt} - \tfrac{1}{2}u_{xx}u_{xtt}, \\ &= \tfrac{1}{2}D_t(u_t u_{xxx}) - \tfrac{1}{2}D_x(u_{xx}u_{tt}). \end{aligned} \tag{3.18}$$

Taking these terms across and including them into the conserved flows, we get

$$D_t(T^1 - \tfrac{1}{2}u_t u_{xxx}) + D_x(T^2 + \tfrac{1}{2}u_{xx}u_{tt}) = 0. \tag{3.19}$$

The modified conserved quantities are now labeled $\tilde{T^i}$, where $D_t(\tilde{T^1}) + D_x(\tilde{T^2}) = 0$ modulo the equation. Then

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_t u_{xxx}, \\ &= \tfrac{1}{2}u_x^2 \\ \tilde{T^2} &= T^2 + \tfrac{1}{2}u_{xx}u_{tt}, \\ &= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t^2 u_x \beta + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2. \end{aligned} \tag{3.20}$$

We have a similar situation below.

(ii) $X = \partial_x, \quad W = -u_x$

The conserved flow is given by $T^1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$ and $T^2 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2\beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$ so that a redefinition leads to

$$\begin{aligned} \tilde{T^1} &= T^1 - \tfrac{1}{2}u_{xx}^2, \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{xxt}, \\ &= -\tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2 \beta + u_x u_{xxt}. \end{aligned} \tag{3.21}$$

Subcase (b): $\alpha \neq 2\beta$. The symmetry generators and conserved vectors are as follows.

(i) $X = \partial_u, \quad B^1 = \frac{1}{2}u_x^2(2\beta - \alpha), \quad B^2 = 0, \quad W = 1$

The conserved flow is given by $T^1 = \frac{1}{2}u_x - \frac{1}{2}\beta u_x^2 - \frac{1}{2}u_{xxx} + \frac{1}{2}u_x^2(2\beta - \alpha)$ and $T^2 = u_x + \frac{1}{2}u_t - u_t u_x \beta - u_{xxt}$ for the total divergence is

$$\begin{aligned} D_t(T^1) + D_x(T^2) &= D_t\left(\tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 - \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}u_x^2(2\beta - \alpha)\right) \\ &\quad + D_x(u_x + \tfrac{1}{2}u_t - u_t u_x \beta - u_{xxt}), \\ &= \tfrac{1}{2}u_{tx} - u_x u_{tx}\beta - \tfrac{1}{2}u_{xxxt} + u_x u_{tx}(2\beta - \alpha) \\ &\quad + u_{xx} + \tfrac{1}{2}u_{tx} - u_x u_{tx}\beta - u_t u_{xx}\beta - u_{xxxt}, \\ &= (u_{xxxt} + \alpha u_x u_{tx} + \beta u_t u_{xx} - u_{tx} - u_{xx}) - \tfrac{1}{2}u_{xxxt}, \\ &= -\tfrac{1}{2}u_{xxxt}. \end{aligned} \tag{3.22}$$

From equation (3.22), u_{xxxt} has two derivative consequences,

$$\begin{aligned} u_{xxxt} &= D_t(u_{xxx}), \\ &= D_x(u_{xxt}), \end{aligned} \tag{3.23}$$

which lead to two possible forms of the same conserved quantity, namely

$$\begin{aligned} \tilde{T}_1^1 &= T^1 + \tfrac{1}{2}u_{xxx}, \\ &= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 + \tfrac{1}{2}u_x^2(2\beta - \alpha) \\ \tilde{T}_1^2 &= T^2, \\ &= u_x + \tfrac{1}{2}u_t - u_t u_x \beta - u_{xxt} \end{aligned} \tag{3.24}$$

or

$$\begin{aligned}\tilde{T}_2^1 &= T^1,\\ &= \tfrac{1}{2}u_x - \tfrac{1}{2}\beta u_x^2 - \tfrac{1}{2}u_{xxx} + \tfrac{1}{2}u_x^2(2\beta-\alpha)\\ \tilde{T}_2^2 &= T^2 + \tfrac{1}{2}u_{xxt},\\ &= u_x + \tfrac{1}{2}u_t - u_t u_x\beta - \tfrac{1}{2}u_{xxt}.\end{aligned} \tag{3.25}$$

(ii) $X = \partial_x, \quad B^1 = \frac{1}{3}u_x^3(2\beta-\alpha), \quad B^2 = 0, \quad W = -u_x$
We get $T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_x^3\beta - \frac{1}{2}u_{xx}^2 - \frac{1}{3}u_x^3(2\beta-\alpha) - \frac{1}{2}u_x u_{xxx}$ and $T^2 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_x^2\beta + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$ so that

$$\begin{aligned}\tilde{T}^1 &= T^1 + \tfrac{1}{2}u_{xx}^2,\\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x^3\beta - \tfrac{1}{2}u_{xx}^2 - \tfrac{1}{3}u_x^3(2\beta-\alpha)\\ \tilde{T}^2 &= T^2 - \tfrac{1}{2}u_x u_{xxt},\\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_t u_x^2\beta + u_x u_{xxt}.\end{aligned} \tag{3.26}$$

3.2.2. Shallow-water wave-2 (SSW-2). For equation (3.13), we use the partial Lagrangian

$$L = \tfrac{1}{2}u_{tx}u_{xx} + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x u_t - \tfrac{1}{2}\beta u_t u_x^2, \tag{3.27}$$

so that

$$\frac{\delta L}{\delta u} = \beta u_{tx}u_x. \tag{3.28}$$

The separation of monomials after substitution into (1.10) gives rise to

$$\begin{aligned}
u_x u_{tx}^2 &: \tau_u,\\
u_{tx}^2 &: \tau_x,\\
u_x u_{xx}^2 &: \xi_u,\\
u_{xx}^2 &: \xi_t,\\
u_{tx}u_{xx} &: \eta_u - \xi_x,\\
u_t u_x u_{tx} &: \beta\tau,\\
u_x^2 u_{tx} &: \beta\xi,\\
u_x u_{tx} &: \beta\eta,\\
u_t u_x^2 &: \xi_x - 3\eta_u,\\
u_t u_x &: \eta_u - \beta\eta_x,\\
u_x^2 &: \eta_u - \tfrac{1}{2}\beta\eta_t - \tfrac{1}{2}\xi_x + \tfrac{1}{2}\tau_t,\\
u_x &: -g_u + \tfrac{1}{2}\eta_t + \eta_x,\\
u_t &: -f_u + \tfrac{1}{2}\eta_x,\\
1 &: f_t + g_x,
\end{aligned} \tag{3.29}$$

from which we clearly need to separate $\beta \neq 0$ or $\beta = 0$.

If $\beta \neq 0$,we have a trivial solution, and if $\beta = 0$, then equation (3.13) changes to

$$u_{xxxt} - u_{tx} - u_{xx} = 0 \tag{3.30}$$

and the partial Lagrangian (3.27) becomes a standard Lagrangian

$$L = \tfrac{1}{2}u_{tx}u_{xx} + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x u_t, \tag{3.31}$$

and the conserved quantities are as follows.

(i) $X = \partial_t, \quad W = -u_t$
The conserved quantities $T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$ and $T^2 = -u_t u_x - \frac{1}{2}u_t^2 + u_t u_{xxt} - \frac{1}{2}u_{xt}^2 - \frac{1}{2}u_{xx}u_{tt}$ lead to a redefinition

$$\begin{aligned} \tilde{T}^1 &= T^1 - \tfrac{1}{2}u_t u_{xxx}, \\ &= \tfrac{1}{2}u_x^2 \\ \tilde{T}^2 &= T^2 + \tfrac{1}{2}u_{xx}u_{tt}, \\ &= -u_t u_x - \tfrac{1}{2}u_t^2 + u_t u_{xxt} - \tfrac{1}{2}u_{xt}^2. \end{aligned} \tag{3.32}$$

(ii) $X = \partial_x, \quad W = -u_x$
Similarly, we obtain $T^1 = -\frac{1}{2}u_x^2 - \frac{1}{2}u_{xx}^2 + \frac{1}{2}u_x u_{xxx}$ and $T^2 = -\frac{1}{2}u_x^2 + u_x u_{xxt} - \frac{1}{2}u_{xx}u_{xt}$ so that

$$\begin{aligned} \tilde{T}^1 &= T^1 - \tfrac{1}{2}u_{xx}^2, \\ &= -\tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_{xx}^2 \\ \tilde{T}^2 &= T^2 - \tfrac{1}{2}u_x u_{xxt}, \\ &= -\tfrac{1}{2}u_x^2 + u_x u_{xxt}. \end{aligned} \tag{3.33}$$

3.3. The regularized long-wave equation

The regularized long-wave equation (RLW) is sometimes referred to as the Benjamin–Bona–Mahoney equation and is shown to possess soliton-type solutions. Solitary waves are wave packets or pulses which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects, these waves retain a stable waveform. A soliton is a very special type of solitary wave, which also keeps its waveform after collision with other solitons. The RLW

$$v_{txx} + \alpha v^2 v_x + v_t + v_x = 0 \tag{3.34}$$

is an alternative description of nonlinear dispersive waves to the more Korteweg–de Vries (KdV) equation. While it is a third-order equation, for our purposes of investigation, we modify this equation, as above, to deal with it in a variational or partial variational way. We refer to the modified RLW, wherein we set $v = u_t$, as RLW-1, namely

$$u_{xxtt} + \alpha u_t^2 u_{tx} + u_{tt} + u_{tx} = 0. \tag{3.35}$$

Alternatively, when $v = u_x$, we get RLW-2 given by

$$u_{xxxt} + \alpha u_x^2 u_{xx} + u_{tx} + u_{xx} = 0. \tag{3.36}$$

3.3.1. Regularized long-wave-1 (RLW-1).

Here, we use the partial Lagrangian

$$L = \tfrac{1}{2}u_{tx}^2 - \tfrac{1}{2}u_t u_x - \tfrac{1}{2}u_t^2 \tag{3.37}$$

for which

$$\frac{\delta L}{\delta u} = -\alpha u_t^2 u_{tx}. \tag{3.38}$$

As before, we obtain Noether-type symmetry generators based on L with the following associated conserved flows which require redefinition so as to fit the PDE. The overdetermined system in question is

$$\begin{aligned}
u_{xx}u_{tx} &: \xi_t + u_t\xi_u,\\
u_{tt}u_{tx} &: \tau_x + u_x\tau_u,\\
u_{tx}^2 &: \eta_u - \tfrac{1}{2}\xi_x - \tfrac{1}{2}\tau_t,\\
u_x u_t^2 u_{tx} &: \alpha\xi,\\
u_t u_t^2 u_{tx} &: \alpha\tau,\\
u_t^2 u_{tx} &: \alpha\eta,\\
u_t u_x u_{tx} &: \eta_{uu},\\
u_t u_{tx} &: \eta_{tu},\\
u_x u_{tx} &: \eta_{xu},\\
u_{tx} &: \eta_{tx},\\
u_x u_t &: \eta_u,\\
u_t^2 &: , \eta_u + \tfrac{1}{2}\xi_x + \tfrac{1}{2}\tau_t\\
u_x &: -g_u - \tfrac{1}{2}\eta_t,\\
u_t &: -f_u - \eta_t - \tfrac{1}{2}\eta_x,\\
1 &: f_t + g_x.
\end{aligned} \tag{3.39}$$

It is clear that if $\alpha \neq 0$, then no symmetry generators are obtained and if $\alpha = 0$, then equation (3.35) becomes

$$u_{xxtt} + u_{tt} + u_{tx} = 0 \tag{3.40}$$

and the partial Lagrangian becomes a standard Lagrangian. In this case, we have the following result.

(i) $X = \partial_t, \quad W = -u_t$

The Noether conserved vector components are $T^1 = -\frac{1}{2}u_{tx}^2 + \frac{1}{2}u_t^2 + u_t u_{txx}$ and $T^2 = \frac{1}{2}u_t^2 + u_t u_{ttx} - u_{tt}u_{tx}$ so that

$$\begin{aligned}
D_t(T^1) + D_x(T^2) &= D_t\left(-\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2 + u_t u_{txx}\right) + D_x\left(\tfrac{1}{2}u_t^2 + u_t u_{ttx} - u_{tt}u_{tx}\right),\\
&= -u_{tx}u_{ttx} + u_t u_{tt} + u_{tt}u_{txx} + u_t u_{ttxx} + u_t u_{tx}\\
&\quad + u_{tx}u_{ttx} + u_t u_{ttxx} - u_{tx}u_{ttx} + u_{tt}u_{txx},\\
&= u_t u_{ttxx} - u_{tx}u_{ttx}.
\end{aligned} \tag{3.41}$$

A redefinition leads to the conserved vector

$$\begin{aligned}
\tilde{T^1} &= T^1 - u_t u_{txx},\\
&= -\tfrac{1}{2}u_{tx}^2 + \tfrac{1}{2}u_t^2\\
\tilde{T^2} &= T^2 + u_{tt}u_{tx},\\
&= \tfrac{1}{2}u_t^2 + u_t u_{ttx}.
\end{aligned} \tag{3.42}$$

(ii) $X = \partial_x, \quad W = -u_x$

We get $T^1 = \frac{1}{2}u_x^2 + u_t u_x + u_x u_{txx} - u_{tx}^2$ and $T^2 = -\frac{1}{2}u_{tx}^2 - \frac{1}{2}u_t^2 + u_x u_{ttx}$ with

$$\begin{aligned}
\tilde{T^1} &= T^1 + u_{tt}u_{tx},\\
&= \tfrac{1}{2}u_x^2 + u_t u_x + u_x u_{txx} - u_{tx}^2 + u_{tt}u_{tx}\\
\tilde{T^2} &= T^2 - u_x u_{ttx},\\
&= -\tfrac{1}{2}u_{tx}^2 - \tfrac{1}{2}u_t^2.
\end{aligned} \tag{3.43}$$

(iii) $X = \partial_u, \quad W = 1$

$T^1 = -\frac{1}{2}u_x - u_t - u_{txx}$ and $T^2 = -\frac{1}{2}u_t - u_{ttx}$ and

$$\begin{aligned} \tilde{T}_1^1 &= T^1 + u_{txx}, \\ &= -\tfrac{1}{2}u_x - u_t \\ \tilde{T}_1^2 &= T^2, \\ &= -\tfrac{1}{2}u_t - u_{ttx} \end{aligned} \tag{3.44}$$

or

$$\begin{aligned} \tilde{T}_2^1 &= T^1, \\ &= -\tfrac{1}{2}u_x - u_t - u_{txx} \\ \tilde{T}_2^2 &= T^2 + u_{ttx}, \\ &= -\tfrac{1}{2}u_t. \end{aligned} \tag{3.45}$$

(iv) $X = t\partial_u, \quad W = t, \quad f = -u, \quad g = -\frac{1}{2}u$

$T^1 = -\frac{1}{2}tu_x - tu_t - tu_{txx} - u$ and $T^2 = -\frac{1}{2}tu_t - tu_{ttx} + u_{tx} - \frac{1}{2}u$ with

$$\begin{aligned} \tilde{T}_1^1 &= T^1, \\ &= -\tfrac{1}{2}tu_x - tu_t - tu_{txx} - u \\ \tilde{T}_1^2 &= T^2 + tu_{ttx}, \\ &= -\tfrac{1}{2}tu_t + u_{tx} - \tfrac{1}{2}u \end{aligned} \tag{3.46}$$

or

$$\begin{aligned} \tilde{T}_2^1 &= T^1 + tu_{txx}, \\ &= -\tfrac{1}{2}tu_x - tu_t - u \\ \tilde{T}_2^2 &= T^2 - u_{tx}, \\ &= -\tfrac{1}{2}tu_t - tu_{ttx} - \tfrac{1}{2}u. \end{aligned} \tag{3.47}$$

3.3.2. Regularized long-wave-2 (RLW-2). Here, we use the partial Lagrangian

$$L = \tfrac{1}{2}u_{xx}u_{tx} - \tfrac{1}{2}u_t u_x - \tfrac{1}{2}u_x^2 \tag{3.48}$$

for which

$$\frac{\delta L}{\delta u} = -\alpha u_x^2 u_{tx} \tag{3.49}$$

and as above $\alpha \neq 0$ leads to no generators $\alpha = 0$ produces the following Noether symmetries and conserved vectors.

(i) $X = \partial_t, \quad W = -u_t$

$T^1 = -\frac{1}{2}u_x^2 + \frac{1}{2}u_t u_{xxx}$ and $T^2 = \frac{1}{2}u_t^2 + u_t u_x + u_t u_{txx} - \frac{1}{2}u_{tx}^2 - \frac{1}{2}u_{tt}u_{xx}$ lead to

$$\begin{aligned} \tilde{T}^1 &= T^1 - \tfrac{1}{2}u_t u_{xxx}, \\ &= -\tfrac{1}{2}u_x^2 \\ \tilde{T}^2 &= T^2 + \tfrac{1}{2}u_{tt}u_{xx}, \\ &= \tfrac{1}{2}u_t^2 + u_t u_x + u_t u_{txx} - \tfrac{1}{2}u_{tx}^2. \end{aligned} \tag{3.50}$$

(ii) $X = \partial_x, \quad W = -u_x$
$T^1 = \frac{1}{2}u_x^2 + \frac{1}{2}u_x u_{xxx} - \frac{1}{2}u_{xx}^2$ and $T^2 = \frac{1}{2}u_x^2 + u_x u_{txx} - \frac{1}{2}u_{tx}u_{xx}$ lead to

$$\begin{aligned} \tilde{T^1} &= T^1 + \tfrac{1}{2}u_{xx}^2, \\ &= \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_x u_{xxx} \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_x u_{txx}, \\ &= \tfrac{1}{2}u_x^2 - \tfrac{1}{2}u_{tx}u_{xx} + \tfrac{1}{2}u_x u_{txx}. \end{aligned} \tag{3.51}$$

(iii) $X = x\partial_u, \quad W = x, \quad f = -\frac{1}{2}u, \quad g = -u$
$T^1 = -\frac{1}{2}xu_x - \frac{1}{2}xu_{xxx} + \frac{1}{2}u_{xx} + \frac{1}{2}u$ and $T^2 = -\frac{1}{2}xu_t - xu_x - xu_{txx} + \frac{1}{2}u_{tx} + u$ so that

$$\begin{aligned} \tilde{T^1} &= T^1, \\ &= -\tfrac{1}{2}xu_x - \tfrac{1}{2}xu_{xxx} + \tfrac{1}{2}u_{xx} + \tfrac{1}{2}u, \\ \tilde{T^2} &= T^2 - \tfrac{1}{2}u_{tx} + \tfrac{1}{2}xu_{txx}, \\ &= -\tfrac{1}{2}xu_t - xu_x - \tfrac{1}{2}xu_{txx} + u. \end{aligned} \tag{3.52}$$

4. Discussion and conclusion

We used the Noether identity to find symmetry generators and then conservation laws for some high-order equations containing mixed derivatives in the highest term. All the conserved vectors in the equations with highest order possessing mixed derivatives produce extra terms that become essential parts of the constructed conserved vector for the equation in question.

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